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1990 J. Phys. A: Math. Gen. 23 841

(http://iopscience.iop.org/0305-4470/23/6/010)

J. Phys. A: Math. Gen. 23 (1990) 841-846. Printed in the UK

Kowalewski's top on the Lie algebras o(4), e(3) and o(3, 1)

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Received 30 November 1988

Abstract. We extend the Lax pair for Kowalewski's top (KT) obtained recently by Haine and Horozov to the Lie algebras o(4) and o(3, 1). The results are expressed in terms of a solution of the Neumann system. We derive formulae for the action variables by analogy with the e(3) KT case and obtain a useful representation for the equations of motion that demonstrate explicitly the separation of variables.

1. Introduction

In the last few years integrable dynamical systems have become very popular. One of the most interesting among them is the Kowalewski top (KT). There exists a large literature devoted to this system [1-7]. A few different Lax representations for the KT and some of its generalisations were constructed [1, 5-7]. It should be emphasised that the Lax representation, in general, is a good starting point for quantisation of a dynamical system.

In order to understand the internal symmetry of the model, it is important to try to analyse any of its integrable deformations or generalisations. In this paper the KT is considered on the Lie algebras o(4), e(3), o(3, 1). Hence, we deal with Kowalewski's basis for the hydrogen atom because one can treat the generators of these three algebras as the angular momentum and the Runge-Lenz vector for the Coulomb field at different energies [3].

Recently, Haine and Horozov [1] reduced the e(3) (standard) KT to the Neumann system, obtaining thus a Lax pair for the top. The 3×3 Lax pair for the Neumann system gives a 3×3 pair for the KT. The spectral curve is equivalent to the hyperelliptic curve used by Kowalewski to integrate the problem in terms of theta functions. Then the Kowalewski equations were derived by means of the standard integration procedure of the Neumann system. It appears that their results can be extended completely to include our cases. The top is reduced to the Neumann system too. We also give the action variables for the KT on o(4), e(3), o(3, 1) and useful representation for the equations of motion in new variables closely connected with the action variables.

2. The generalised Kowalewski top and the Neumann system

Let us consider a classical dynamical system on the orbits of the Lie algebra $\mathscr{E}$ with generators J_i, x_i, $i=1,2,3$, obeying the following Poisson brackets ($\mathscr{P}=$ constant)

$$\begin{aligned} &\{J_i, J_j\}=\varepsilon_{ijk}J_k \qquad \{J_i, x_j\}=\varepsilon_{ijk}x_k \\ &\{x_i, x_j\}=-\mathscr{P}\varepsilon_{ijk}J_k. \end{aligned} \tag{1}$$

We distinguish the orbits by fixing the values of the Casimir elements

$$l=\sum_i J_i x_i \qquad a^2=\sum_i (x_i x_i-\mathscr{P}J_i J_i). \tag{2}$$

For the special cases $\mathscr{P}=0, \pm 1$ $\mathscr{E}$ is the Lie algebra of the Euclid group E(3), the group of four-dimensional compact rotations O(4) and the Lorentz group O(3, 1) respectively:

$$\mathscr{E}=\begin{cases} \mathrm{o}(4) & \mathscr{P}=-1 \\ \mathrm{e}(3) & \mathscr{P}=0 \\ \mathrm{o}(3,1) & \mathscr{P}=1. \end{cases} \tag{3}$$

The Hamiltonian of the KT on the orbits of the algebra $\mathscr{E}$ is ($b=$ constant)

$$H=J_1^2+J_2^2+2J_3^2-2bx_1. \tag{4}$$

The dynamical equations are determined by the rule

$$\frac{\mathrm{d}}{\mathrm{d}t}=\tfrac{1}{2}\{H, \cdot\}. \tag{5}$$

The additional integral of motion that commutes with H has the form

$$\begin{aligned} &K=k_+k_- \qquad k_\pm=J_\pm^2+2bx_\pm-\mathscr{P}b^2 \\ &J_\pm=J_1\pm \mathrm{i}J_2 \qquad x_\pm=x_1\pm \mathrm{i}x_2 \end{aligned} \tag{6}$$

$$\{H, K\}=0. \tag{7}$$

Thus we consider the integrable system on the orbits of the algebra $\mathscr{E}$ given by two integrals of motion (4) and (6). For the special case $\mathscr{P}=0$ we have the standard KT. For further generalisations of the system to the gyrostat and its quantum counterparts see [3, 4, 8].

Following [1], we now proceed to the reduction of the top to Neumann's system. New variables

$$\begin{aligned} &z_1=\frac{1}{J_+-J_-} \qquad z_2=\frac{J_++J_-}{J_+-J_-} \qquad z_3=\frac{J_+J_-}{J_+-J_-} \\ &z_4=\frac{J_3}{J_+-J_-} \qquad z_5=\frac{-bx_3}{J_+-J_-} \qquad z_6=\frac{J_3J_+J_-+bx_3(J_++J_-)}{J_+-J_-} \end{aligned} \tag{8}$$

are introduced. They satisfy the quadratic relations:

$$\begin{aligned} &q_1=z_2^2-4z_1z_3=1 \\ &q_2=z_1z_6+z_2z_5-z_3z_4=0 \\ &q_3=(H+2\mathscr{P}b^2)z_1z_3+2blz_1z_2+2b^2\tilde{a}z_1^2-\tfrac{1}{2}\mathscr{P}b^2z_2^2-2z_5^2-\tfrac{1}{2}z_3^2-2z_4z_6=0 \\ &q_4=8b^2z_1^2[(H+\mathscr{P}b^2)\tilde{a}-2l^2]+(H+\mathscr{P}b^2)(4\mathscr{P}b^2z_1z_3-2z_3^2-8z_5^2)-4b^2\tilde{a}(4z_4^2+z_2^2) \\ &\qquad -8bl(z_2z_3+4z_4z_5-\mathscr{P}b^2z_1z_2)+4z_6^2-\mathscr{P}^2b^4z_2^2-8\mathscr{P}b^2z_4z_6=K \end{aligned} \tag{9}$$

where we denote

$$\tilde{a} = a^2 - K/4b^2 + \mathcal{P}H/2 + b^2\mathcal{P}^2/4.$$

The dynamical equations for $z = (z_1, \ldots, z_6)'$ have the form

$$\dot{z} = \tfrac{1}{2}\mathrm{i}M\nabla_z q_3 \tag{10}$$

where

$$M = \begin{pmatrix} 0 & F \\ -F' & G \end{pmatrix} \qquad G = \begin{pmatrix} 0 & z_4 & -2z_5 \\ -z_4 & 0 & z_6 \\ 2z_5 & -z_6 & 0 \end{pmatrix} \qquad F = \begin{pmatrix} 0 & z_1 & -z_2 \\ -2z_1 & 0 & -2z_3 \\ -z_2 & -z_3 & 0 \end{pmatrix}. \tag{11}$$

From the z_k, we go further to new variables ρ_i, l_i $(i = 1, 2, 3)$ defined as [1]

$$\begin{aligned} &\rho_1 = (z_1 + z_3)/2 \qquad &&\rho_2 = z_2/2i \qquad &&\rho_3 = (z_1 - z_3)/2i. \\ &l_1 = (z_4 - z_6)/2 \qquad &&l_2 = z_5/i \qquad &&l_3 = (z_4 + z_6)/2i \end{aligned} \tag{12}$$

and postulate new Poisson brackets of the Lie algebra e(3) for ρ_i, l_i:

$$\begin{aligned} &\{l_i, l_j\}' = \varepsilon_{ijk} l_k \qquad \{l_i, \rho_j\}' = \varepsilon_{ijk}\rho_k \\ &\{\rho_i, \rho_j\}' = 0. \end{aligned} \tag{13}$$

The Casimir elements of the algebra e(3) (13) are

$$\sum_i l_i\rho_i = -q_2/2 = 0 \qquad \sum_i \rho_i^2 = -q_1/4 = -\tfrac{1}{4} \tag{14}$$

where (9) are employed. Obviously the primed brackets (13) cannot be derived from the old ones (1).

The dynamical equations for the vector $\boldsymbol{\xi} = (\rho_1, \rho_2, \rho_3, l_1, l_2, l_3)'$ are

$$\dot{\xi} = \tfrac{1}{2}\mathrm{i}T(\boldsymbol{\xi})\nabla_{\xi} q_3 \tag{15}$$

where

$$T(\xi) = \begin{pmatrix} 0 & P \\ P & L \end{pmatrix} \qquad P = \begin{pmatrix} 0 & \rho_3 & -\rho_2 \\ -\rho_3 & 0 & \rho_1 \\ \rho_2 & -\rho_1 & 0 \end{pmatrix} \qquad L = \begin{pmatrix} 0 & l_3 & -l_2 \\ -l_3 & 0 & l_1 \\ l_2 & -l_1 & 0 \end{pmatrix}. \tag{16}$$

Using the explicit form of q_3, the system (15) is

$$\dot{\boldsymbol{\rho}} = 2\mathrm{i}\, \boldsymbol{l}\wedge\boldsymbol{\rho} \qquad \dot{\boldsymbol{l}} = 2\mathrm{i}\, Q\boldsymbol{\rho}\wedge\boldsymbol{\rho} \tag{17}$$

where the matrix $Q = Q'$

$$Q = \begin{pmatrix} -\frac{1}{4} + b^2\tilde{a} & \mathrm{i}bl & i(\frac{1}{4} + b^2\tilde{a}) \\ \mathrm{i}b\,l & -H/2 & -bl \\ \mathrm{i}(\frac{1}{4} + b^2\tilde{a}) & -bl & \frac{1}{4} - b^2\tilde{a} \end{pmatrix} \tag{18}$$

depends only on constants of the motion and is thus itself constant. Equations (17) are a direct consequence of the changes of variables $J_i, x_i \to z_1, \ldots, z_6 \to \rho_i, l_i$ at the fixed values of the Casimir elements (2) and the constants of motion H, K (4) and (6). On the other hand, (17) can be treated as Hamiltonian equations on the orbits of the auxiliary algebra e(3) given by equations (13) and (14) with a new Hamiltonian

$$\mathrm{i}q_3 = \mathrm{i}(2\boldsymbol{l}^2 + 2(Q\boldsymbol{\rho}, \boldsymbol{\rho}) - \tfrac{1}{2}(H/2 + \mathcal{P}b^2)) = 0 \tag{19}$$

according to the rule $d/dt = \frac{1}{2}(iq_3, \cdot\}'$. An additional integral of motion in involution with q_3 (H = constant) is

$$q_4 = -16(Ql, l) + 16 \det Q(Q^{-1}\boldsymbol{\rho}, \boldsymbol{\rho}) + \mathcal{P}^2 b^4 = K. \tag{20}$$

Thus we have connected the generalised KT to the integrable Neumann system with the special values of the constants of motion according to (14), (19) and (20) that are expressed in terms of the Hamiltonian and the additional integral of the KT. A Lax representation is known for the Neumann system [9]. From this follows a Lax pair for the generalised KT

$$\dot{\mathcal{L}}(u) = [\mathcal{L}(u), \mathcal{A}(u)] \tag{21}$$

where $[\cdot, \cdot]$ is a matrix commutator,

$$\mathcal{L}(u) = Q + Lu - Nu^2 \qquad \mathcal{A}(u) = -2i(Qu^{-1} + L) \tag{22}$$

$u \in \mathbb{C}$ is a spectral parameter, Q and L are defined by (18) and (16), respectively, and the matrix N is equal to

$$N = \rho \otimes \rho \qquad N_{ij} = \rho_i \rho_j. \tag{23}$$

Equation (21) is a Lax representation (with a spectral parameter) of (17) and, hence, of the initial equations for the generalised KT too.

The spectral curve Γ: $\det(\mathcal{L}(u) - \lambda I) = 0$, where $q_2 = 0$ is assumed according to (9), is a hyperelliptic genus-2 curve:

$$\begin{aligned} &u^2[(\lambda - \mathcal{P}b^2)^2/4 - K/16] - P_3(\lambda) = 0. \\ &P_3(\lambda) = (\lambda + H/2)(\lambda^2 + b^2\tilde{a}) - b^2 l^2. \end{aligned} \tag{24}$$

For $\mathcal{P} = 0$ the curve (24) is equivalent to the familiar Kowalewski curve [1].

3. Separated equations and action variables

Let us consider the intersection of the two curves Γ and $\tilde{\Gamma}$:

$$\begin{cases} \det(Q + Lu - Nu^2 - \lambda I) = 0. \\ \det(Q + Lu - \lambda I) = 0. \end{cases} \tag{25}$$

Notice that $\tilde{\Gamma}$ is a curve associated with the Euler top for which the Lax matrix in (21) is $\tilde{\mathcal{L}} = Q + Lu$. From (25) the equation for λ follows:

$$\lambda^2 + \lambda(H/2 - 4(Q\boldsymbol{\rho}, \boldsymbol{\rho})) - 4 \det Q(Q^{-1}\boldsymbol{\rho}, \boldsymbol{\rho}) = 0. \tag{26}$$

It is easy to see that the quadratic equation (26) in the case $\mathcal{P} = 0$ coincides with the definition of the separated variables for the standard KT. Further, one can always diagonalise the matrix $Q \to \tilde{Q} = \operatorname{diag}(a_1, a_2, a_3) = RQR^{-1}$ by the corresponding rotation R of the vectors $\boldsymbol{\rho}$ and $\boldsymbol{l}$: $\tilde{\boldsymbol{\rho}} = R\boldsymbol{\rho}$, $\tilde{\boldsymbol{l}} = R\boldsymbol{l}$. Then (26), divided by $\det(\tilde{Q} - \lambda I) = (\lambda - a_1)(\lambda - a_2)(\lambda - a_3)$, becomes

$$\frac{\tilde{\rho}_1^2}{\lambda - a_1} + \frac{\tilde{\rho}_2^2}{\lambda - a_2} + \frac{\tilde{\rho}_3^2}{\lambda - a_3} = 0. \tag{27}$$

Now the dynamics of two roots λ_1, λ_2 of (26) can be derived by the integration procedure of the Neumann system [1]

$$\varepsilon_i \dot{\lambda}_i(\lambda_1 - \lambda_2) = 2(-R_5(\lambda_i))^{1/2} \qquad \varepsilon_i = (-1)^{i+1}$$
$$R_5(\lambda) = P_2(\lambda)P_3(\lambda) \tag{28}$$
$$= [(\lambda - \mathscr{P}b^2/2)^2 - K/4][(\lambda + H/2)(\lambda^2 + b^2\tilde{a}) - b^2l^2].$$

Equations (28) generalise the familiar Kowalewski equations.

To obtain action variables we have to restore the Hamiltonian structure of (28) in the same way as we did it in the e(3) case [2]. As a result the Lagrange variables λ_i, $\dot{\lambda}_i$ turn into the Hamiltonian ones s_i, p_i, $i = 1, 2$,

$$s_i = 2\lambda_i + H \qquad p_i = \frac{1}{2(-2s_i)^{1/2}} \ln\{[x_i + (x_i^2 + d_i^2)^{1/2}]/d_i\}$$
$$x_i = 2(y_i^2 + d_i y_i)^{1/2} \qquad y_i = (s_i - \mathscr{P}b^2 - H)^2 - K \tag{29}$$
$$d_i = 4b^2(a^2 + \mathscr{P}s_i/2 - 2l^2/s_i).$$

Then the action variables have the form

$$S_i = \oint_{\alpha_i} p(s)\, \mathrm{d}s \tag{30}$$

where α_i are α-cycles of the Jacobi variety of the algebraic curve $w^2 = -R_5(s)$, where $R_5(s)$ is defined by (28). The action is a sum of the two items S_1 and S_2 depending on s_1 and s_2, respectively. So we have a separation of variables.

In terms of the Hamiltonian variables s_i, p_i, $i = 1, 2$, (28) look like

$$s_i^3 - (2H + \mathscr{P}b^2)s_i^2 + \varkappa s_i - 4b^2l^2 = 2b^2(a^2 s_i + \mathscr{P}s_i^2/2 - 2l^2)\cos[2(2s_i)^{1/2}p_i]$$
$$\varkappa = (H + \mathscr{P}b^2)^2 - K + 2b^2a^2 \tag{31}$$

that demonstrate explicitly the separation of variables.

Like the e(3) case we rewrite the separated equations (31) in terms of generators of the direct sum of two Lie algebras of rank 1. Instead of p_i, s_i let us introduce new variables:

$$u_i = (s_i)^{1/2} \qquad m_i^{\mp} = \exp(\pm \mathrm{i}22^{1/2}u_i p_i) \tag{32}$$

that obey e(2)⊕e(2) Poisson brackets

$$\{u_i, m_k^{\mp}\} = \mp \mathrm{i}2^{1/2} m_k^{\mp}\delta_{ik}$$
$$\{m_i^+, m_k^-\} = 0 \qquad \{u_i, u_k\} = 0. \tag{33}$$

The invariant Casimir elements are equal to $m_k^+ m_k^- = 1$. Equations (31) turn into $(i = 1, 2)$

$$u_i^6 - (2H + \mathscr{P}b^2)u_i^4 + \varkappa u_i^2 - 4b^2l^2 = b^2(a^2u_i^2 + \mathscr{P}u_i^4/2 - 2l^2)(m_i^+ + m_i^-). \tag{34}$$

These equations have the form typical for the R matrix method [10, 11].

Having the action variables (30) one can consider quasiclassical quantisation of the generalised KT as in the e(3) case [2].

4. Discussion

Our results are closely connected with quantisation of the KT. The integrals of motion for the quantum KT are known [3], so by quantisation we mean the way of calculating their spectrum. Quasiclassical quantisation of the KT was carried out recently [2] and using formulae given in section 3 these results can be easily repeated for the algebras o(4) and o(3, 1). In contrast, the corresponding procedure for the quantum mechanics case is not yet known. In principle, such a procedure must be a consequence of a Lax representation. For the KT four Lax pairs are known [1, 5-7]. The Lax pair of Perelomov [6] does not depend on a spectral parameter. As we proved here the Lax pair of Haine and Horozov [1] admits a generalisation to the o(4) and o(3, 1) algebras, but non-canonical transformation to new variables is needed. Adler and van Moerbeke [7] introduced a Lax pair that connected the KT and the Manakov top. It is also based on a non-canonical transformation.

The Lax pair of Reyman and Semenov-Tian-Shansky [5] seems to be the best candidate for quantisation, though it is not clear how to extend it to the algebras o(4) and o(3, 1).

Acknowledgments

One of authors (IVK) is grateful to Rolf Tarrach for his hospitality at the Universitat de Barcelona and to the cooperative agreement between the Universities of Leningrad and Barcelona which made the stay in Barcelona possible.

References

[1] Haine L and Horozov E 1987 *Physica* D **29** 173
[2] Komarov I V and Kuznetsov V B 1987 *Teor. Mat. Phys.* **73** 335
[3] Komarov I V 1981 *Teor. Mat. Phys.* **47** 67
[4] Komarov I V 1987 *Phys. Lett.* **123A** 14
[5] Reyman A G and Semenov-Tian-Shansky M A 1987 *Lett. Math. Phys.* **14** 55
[6] Perelomov A M 1981 *Commun. Math. Phys.* **84** 239
[7] Adler M and Moerbeke P 1988 *Commun. Math. Phys.* **113** 659
[8] Yahia H M 1987 *Vestnik MGU, Ser. 1* **4** 88
[9] Perelomov A M 1981 *Funct. Anal. Pril.* **15** 83
[10] Sklyanin E K 1985 *J. Sov. Math.* **31** 3417
[11] Kuznetsov V B and Tsiganov A V 1989 *J. Phys. A: Math. Gen.* **22** L73-9